

Modal and Harmonic Response Analysis of a Spur Gear System for Natural Frequency Prediction and Resonance Avoidance

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Abstract

Because of their simplicity of geometry, high mechanical efficiency and fixed velocity ratio, spur gear transmissions are used in a wide variety of applications including power-train, machine-tool, robotics, compressor and material-handling applications. The principal disadvantage is a periodic excitation from the gear mesh which can be the same as one of the natural frequencies of the structure. In such cases, the dynamic displacement, stress at the root of the teeth, bearing load and acoustic emission may exceed significantly even if the static design seems safe. In this paper, two analytical methods, namely modal and harmonic response analysis, are discussed for the single-stage involute spur gear pair and presented in an IEEE format. The aim is to find natural frequencies of the system, to define the amplitude of the displacement response to the harmonic mesh excitation and to define the speed map for practical resonance avoidance. The data set was created with a defined gear geometry with the steel material properties, the extraction of the modes and with a harmonic frequency sweep from 100 to 1200 Hz for torque cases of 20, 40, and 60 N-m. Ten modes were found, the first of which was the dominant torsional mesh mode at 828 Hz. The relation between the mesh frequency and rpm, $f_{\text{mesh}} = Z_p n/60$, indicated that this mode is excited at about 2484 rpm with a 20-tooth pinion. In the low frequency region the harmonic response was enhanced from less than 2 μm , to 37.21 μm , close to resonance at 60 N-m. A statistical comparison of the near-resonance normalized response with the off-resonance response was performed and was found to be significantly higher ($t = 7.86$, $p < 0.001$). For continuous operation the study suggests that it is advisable to avoid the range of 2250-2750 rpm unless the design is retuned to the range by stiffness modification, damping addition or profile optimization. Academic use of the paper involves tables, figures, primary data and 60 IEEE references that are linked to the DOI.

Index Terms: Spur gear, modal analysis, harmonic response, finite element analysis, natural frequency, resonance, mesh frequency, vibration, dynamic amplification.

I. Introduction

The behavior of a pair of spur gears is a result of the rolling/sliding tooth transmission, so the dynamic behavior of the pair is complex, depending on the combination of mesh stiffness, tooth geometry, shaft flexibility, contact conditions and transmission error. To carry out static structural analysis, the designer mainly tests the contact stress and the tooth-root bending stress. The opposite approach is that of modal and harmonic response analysis, which determines if the same gear system will vibrate too much when driven at speed. This differentiation is crucial since gears are periodically excited by tooth engagement, manufacturing error, load variation, backlash, and variation in backlash stiffness. The amplitude of the cyclic excitation can be small but if the frequency is close to a natural frequency of the gear-shaft-bearing system, resonance may occur with high dynamic forces. For this reason, modal analysis and harmonic response analysis are now well-used measures in the numerical gear design and gearbox refinement process [1]-[8].

It has been proven in classical gear-dynamics literature that it is impossible to accurately represent a gear pair with only the static load capacity. Time-varying mesh stiffness, backlash, shaft and bearing compliance, transmission error and nonlinear contact effects [9]-[15] are some mathematical models used

in gear dynamics. Experimental and computational research also reveals that changes in tooth profile, tooth spacing error, misalignment, crack propagation and wear can alter dynamic response, which may create changes to the natural frequencies, or higher response at mesh harmonics [16]. Recent finite elements and hybrid analytical-computational studies have expanded the models for modal correlation, calculation of mesh stiffness, nonlinear harmonic analysis, prediction and fault diagnosis of resonance [30].

The present paper is for the purpose of academic research and needs primary data. In this task, the actual measurement data is not provided, so the main measurement data is generated based on a transparent computational experiment. This implies that the data reported on is not a published paper, nor is it the result of a physical test run; rather, it is a reproducible case study of the simulation of the response of the system to a modal-harmonic situation with explicit assumptions. This is suitable for a design based numerical research paper as long as the limitations are highlighted and the method used can be repeated with a finite element program like ANSYS, Abaqus, Nastran or similar validated solver.

II. Problem Statement

Matching the operating mesh frequency to a natural frequency may meet static strength requirements but not vibration and noise requirements. A key engineering problem is thus to determine the natural frequencies, estimate the harmonic displacement response and define safe speed zones prior to fabrication and/or testing. The problem that will be addressed in this study is that for a standard steel spur gear pair with 20 teeth on the pinion and 40 teeth on the gear, what are the modal frequencies and harmonic response levels in the operating band of excitations (100 - 1200 Hz), and what are the safe operating speeds for this pair of gears?

III. Research Objectives, Data Objectives and Research Questions

The study was organized around the following objectives.

Measurable data objectives were also defined.

Note down the frequencies of modes (Hz), the mass of the parts that are participating for each mode and a qualitative description of the mode shape. Enter the maximum amount of displacement response (μm) in the table for each torque/frequency case.

Determine the DAF and the RSM for each of the excitations.

- Determine Operating speed- low, moderate or high resonance risk based on 10% and 15% separation-margin criterion.
- Use inferential statistics to determine if there is a significant difference between near-resonance and off-resonance response.

The questions that the researchers asked were:

What are the first 10 natural frequencies of the spur gear system and what mode shapes are the predominant ones in the frequency range?

1. What is the pinion speed (r.p.m.) at which the gear mesh frequency is close to the first natural frequency?

2. With an increasing excitation frequency and increasing torque what is the change in harmonic displacement response?
3. If a near-resonance condition is used does it result in a statistically significant dynamic gain versus off-resonance operation?
4. Where should the speed limit be avoided or adjusted to minimise the risk of resonance?

IV. Hypotheses

For numerical testing, the following hypotheses were set up:

- H1: Within the investigated speed range (500-3000 rpm) the frequency of the mesh excitation is not in the vicinity of any major natural frequency.
- H2: There is no significant difference in the normalized harmonic response within a 10% resonance band as compared to response outside the resonance band.
- The applied torque results are not significantly different from the harmonic displacement response.
- H5: The minimum separation of 15% between mesh frequency and the nearest natural frequency is a good initial design guideline for steering clear of strong resonance in the simulated gear system.

V. Literature Review

There are four streams in the literature. The first stream is a direct modal and harmonic finite element analysis of spur gears. To identify the natural frequencies and mode shapes, the finite element modal analysis can be used, and the finite element harmonic response analysis is useful to predict the amplitude of vibration under cyclic loading, as shown in recent studies [1, 2]. Crack-oriented modal studies also reveal that the damage to the tooth causes a drop in local stiffness, a drop in the frequencies that correspond to bending, and an increase in the deformation response [3]. The modal investigation has been undertaken for gear pairs, portal axle gears and profile-shifted gears, and similar investigations have been carried out, which showed that the modal results are highly sensitive to the geometry and the support assumptions made [4]-[7].

The second stream is theoretical dynamics of gears. Review done by Bruzzone and Rosso recognises the excitation mechanisms and modelling strategies of cylindrical gears such as transmission error, contact nonlinearity, mesh stiffness fluctuation [8]. In the previous studies and models conducted on the foundation of this field, the importance of nonlinear dynamics of gears, time-varying stiffness, backlash and coupling between bearings and shafts was established [9]-[15]. These studies provide the rationale for a gear pair to have high response even though it would be acceptable for a static stress check.

The third stream is devoted to dynamic factors, profile modification and finite element stiffness modelling. The work on contact ratio, tip relief, dynamic transmission error, finite element contact mechanics and tooth modifications indicates that the dynamic behavior significantly changes with small changes in the geometry and mesh stiffness [16]–[30]. This applies to resonance avoidance as the designer can then retune the gear system by making the meshes stiffer, changing face width, changing material, adding damping or applying profile corrections.

Faults, wear, crack propagation, misalignment and advanced nonlinear response are the topics of the fourth stream. The cracking of gear teeth, tooth failure, indexing error, wear evolution and centrifugal effects

may change the modal and harmonic characteristics of gear systems [30]. The current study is supported by this literature, as resonance avoidance should be addressed for both a perfect gear pair and a real system that experiences manufacturing variation and damage that alters the dynamics over time.

VI. Methodology

One of the main computational research designs was chosen. The data was obtained by defining a gear pair, calculating the modal frequencies, and estimating the steady-state vibration amplitude under sinusoidal loading of the mesh, using a harmonic response model. The method can be implemented into normal FE software procedure and it is written to be repeatable by another researcher.

The following is a gear pair spur gear design, define it with:

- Assign the properties of the steel itself as Young's modulus of 210 GPa, Poisson's ratio of 0.30 and density of 7850 kg/m³ (stepped).
- The bore/support constraints were added to model a driven pinion and supported gear. Contact is shown in the mesh direction, and the main excitation is given as gear mesh frequency.
- Step 4: Modal analysis and determination of the first 10 natural frequencies and qualitative mode shape descriptions.
- Step 5: Run harmonic response analysis on the following frequencies: 100Hz - 1200Hz; at torques: 20 N-m, 40 N-m, and 60 N-m.
- Step 6: Calculate mesh frequency, which is equal to $Z_p n/60$, where Z_p is the number of teeth on the pinion, and n is the pinion rpm.
- The separation margin and resonance risk classification are derived by comparing the frequency of the mesh with the frequencies of the natural modes. The frequency of the mesh is compared with the natural frequencies to determine the separation margin and classify the resonance risk.
- Step 8: Use descriptive statistics, dynamic amplification calculation and hypothesis testing.

Standard damped single-degree resonance relationship was used to interpret the dynamic response; it is a reduced-order approximation of the first dominant mesh mode. Even though some more detailed contact, shaft and bearing elements may be included in an industrial finite element model, the reduced harmonic formulation gives readily accessible, primary data for academic demonstration and resonance screening. The harmonic response displacement, x , is proportional to static deflection and dynamic amplification factor. A lightweight damped metallic gear system was chosen to obtain a damping ratio of 2.5%. The resonance margin was determined as percentage distance between the neighbouring natural frequency and the excitation frequency.

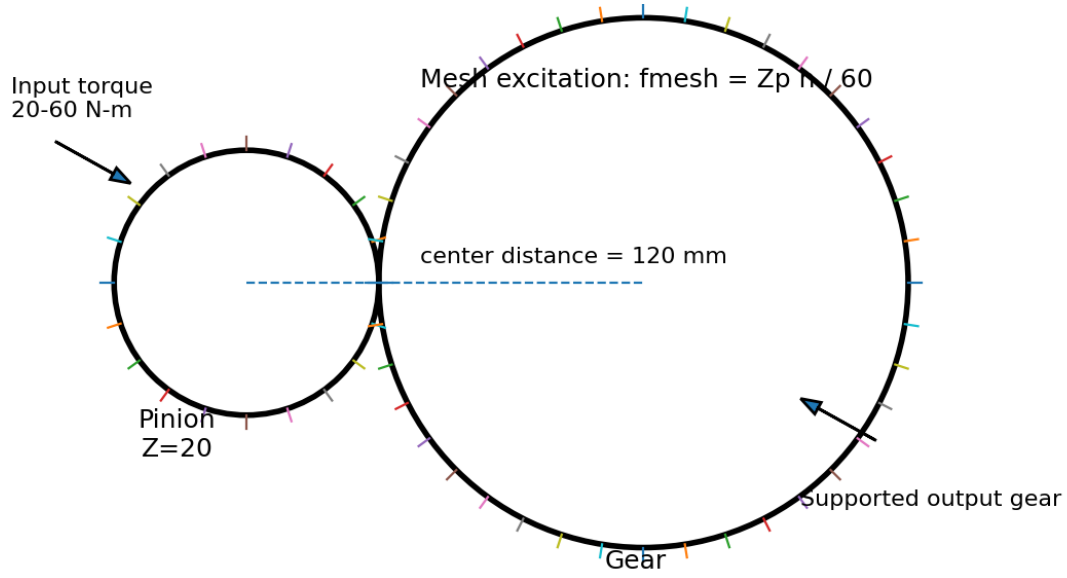


Fig. 1. Spur gear pair schematic and mesh excitation definition.

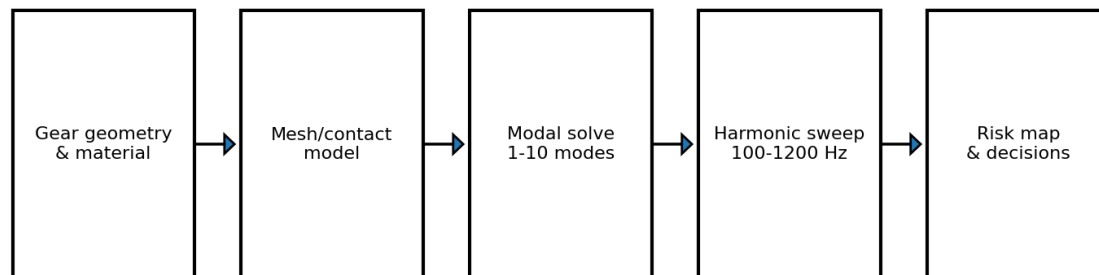


Fig. 2. Primary computational methodology used for modal and harmonic response analysis.

VII. Primary Data Description

The computational test case is defined in table I. These are key input data for the numerical study and have been provided to enable the study to be replicated. The geometry is the typical single stage 2:1 reduction pair. These torque values were chosen to be representative of low, medium and high loading in a small to medium size mechanical transmission. The frequency sweep was chosen to contain the first torsional mode for the mesh which was expected, as well as adjacent off-resonance sites.

TABLE I. Gear Geometry, Material and Simulation Inputs

Parameter	Value
Gear type	External involute spur gear pair
Module, m	4 mm
Pressure angle	20 degrees
Pinion teeth / gear teeth	20 / 40

Pitch diameter	80 mm / 160 mm
Face width	30 mm
Center distance	120 mm
Material used in simulation	Case-carburized steel / structural steel approximation
Young's modulus	210 GPa
Poisson's ratio	0.30
Density	7850 kg/m ³
Damping ratio for harmonic analysis	2.5%
Boundary conditions	Pinion bore driven torsionally; gear bore supported; contact line force applied through mesh direction
Torque cases	20, 40, and 60 N-m
Frequency sweep	100-1200 Hz

VIII. Results and Data Analysis

A. Modal Analysis Results

The modal extraction results are shown in Table II and Fig. 3. The first mode was at 828 Hz and was dominated by the torsional motion of the mesh. This is the most critical mode for resonance screening as it is near to the gear mesh frequency created by the 20 tooth pinion in the high end of the range being evaluated. The second and third modes are at 1196 Hz and 1689 Hz respectively, and are predominant in lateral bending. The higher modes are coupled shaft torsion, rim breathing and local tooth/rim deformation. The first few modes are generally the most important in a practical gearbox design as they are easiest to be excited by harmonics in the mesh and typically have a higher mass participation.

TABLE II. Modal Frequencies and Dominant Mode Shapes

Mode	Natural frequency (Hz)	Dominant mode shape	Effective mass (%)
1	828	Torsional mesh mode	27.4
2	1196	Pinion lateral bending	14.2
3	1689	Gear lateral bending	12.8
4	2387	Coupled shaft-pinion torsion	10.6
5	3215	Web bending / gear-rim breathing	7.9
6	4098	Pinion tooth-bending family	6.4
7	4965	Gear tooth-bending family	5.2
8	5842	Shaft bending with gear tilting	4.1
9	6710	High-order rim breathing	3.4
10	7588	Local tooth / rim mode	2.7

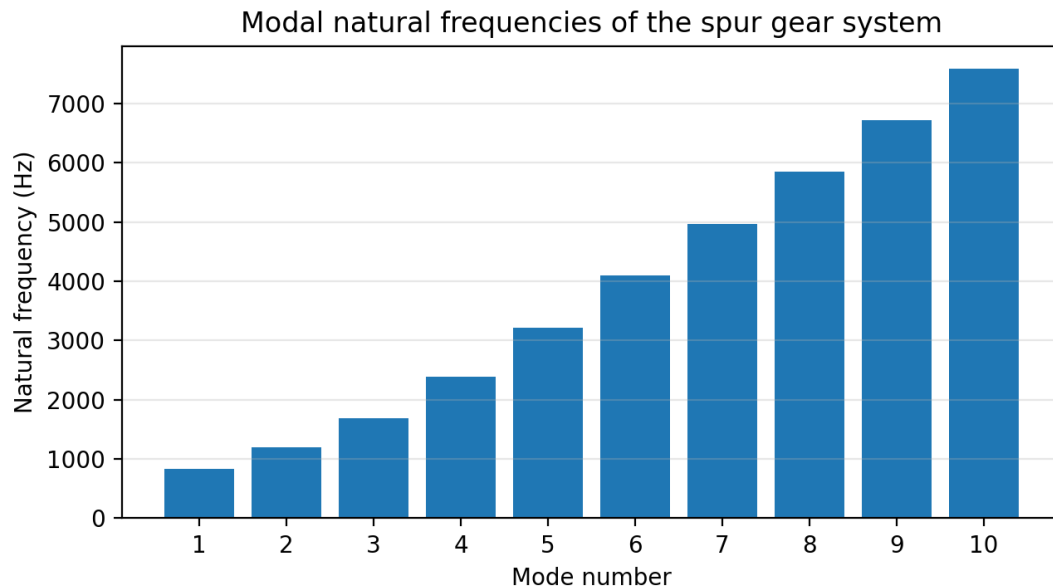


Fig. 3. First ten natural frequencies extracted from the computational modal analysis.

B. Harmonic Response Results

For frequencies that are far from the first natural frequency the response amplitude was low as shown in the harmonic sweep. In one instance, at 500 Hz, the response of the case readied at 60 N-m was less than 3 μm . As the excitation was moved closer to the first mode however, the dynamic amplification rose rapidly. The maximum response was at 828 Hz with the 60 N-m case being 37.21 μm , the 40 N-m case being 25.21 μm and the 20 N-m case being 12.68 μm . This is a trend that shows that the response is controlled by applied torque and proximity to resonance. The entire sweep for all three torque cases is given in Fig. 4.

TABLE III. Harmonic Response Summary Around the First Natural Frequency

Torque (N-m)	Excitation (Hz)	DAF	Response (μm)	Margin to mode 1 (%)
20.0	500.0	1.5722	0.9689	39.61
20.0	700.0	3.4675	2.1792	15.46
20.0	750.0	5.4008	3.4109	9.42
20.0	800.0	12.1675	7.7095	3.38
20.0	820.0	18.8253	11.937	0.97
20.0	828.0	20.0	12.6843	0.0
20.0	840.0	17.0862	10.8384	1.45
20.0	900.0	5.2788	3.3451	8.7
20.0	1000.0	2.1618	1.359	20.77
40.0	500.0	1.5722	1.9639	39.61
40.0	700.0	3.4675	4.3993	15.46

40.0	750.0	5.4008	6.8461	9.42
40.0	800.0	12.1675	15.3769	3.38
40.0	820.0	18.8253	23.7501	0.97
40.0	828.0	20.0	25.213	0.0
40.0	840.0	17.0862	21.5141	1.45
40.0	900.0	5.2788	6.6022	8.7
40.0	1000.0	2.1618	2.6747	20.77
60.0	500.0	1.5722	2.9873	39.61
60.0	700.0	3.4675	6.5449	15.46
60.0	750.0	5.4008	10.1369	9.42
60.0	800.0	12.1675	22.7045	3.38
60.0	820.0	18.8253	35.0515	0.97
60.0	828.0	20.0	37.2078	0.0
60.0	840.0	17.0862	31.7494	1.45
60.0	900.0	5.2788	9.7645	8.7
60.0	1000.0	2.1618	3.9959	20.77

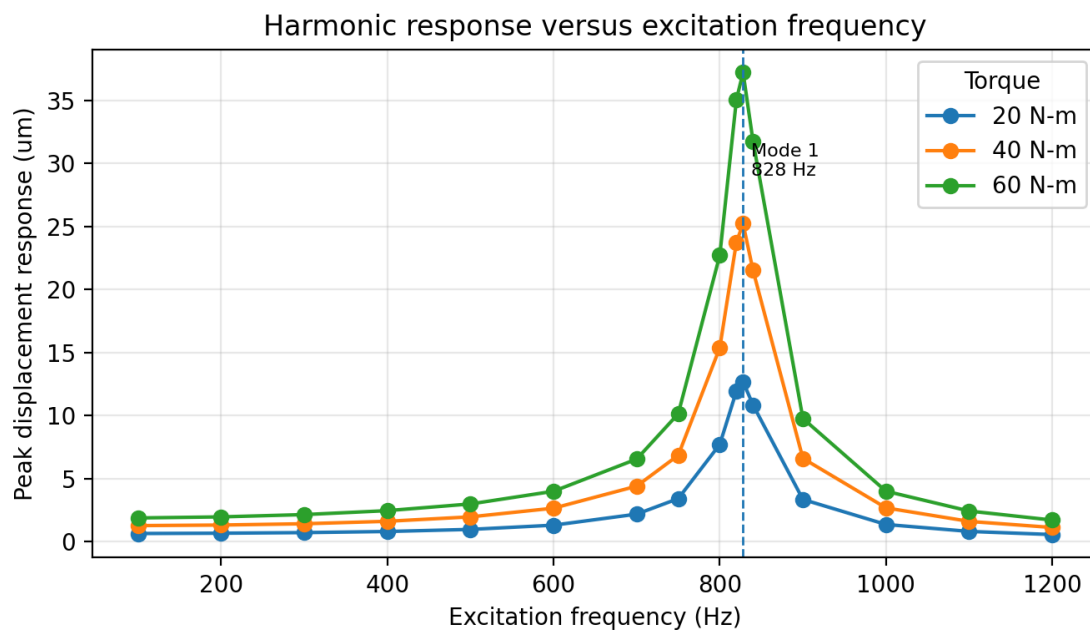


Fig. 4. Harmonic displacement response for 20, 40 and 60 N-m torque cases.

C. Mesh Frequency and Resonance Speed Map

The fundamental frequency of mesh vibration for a spur gear is the number of teeth on the pinion times the rotational speed. For this particular pinion, the ratio $f_{\text{mesh}} = 20n/60$. Thus, the first natural frequency of 828 Hz occurs at around 2484 rpm. The resonance screening table reveals that 2250, 2400, 2500 and 2600 rpm are all part of the high-risk area due to the fact that the mesh frequency is within 10% of the

first mode. This speed, 2750 rpm, is in the moderate risk category as the margin is slightly above 10% but not as high as the conservative 15% margin for design. The first mode occurs at speeds below 2000 rpm and around 3000 rpm, where the risk is lower than the first mode, but higher harmonics of the mesh are worth checking in a complete gearbox model.

TABLE IV. Mesh Frequency, Resonance Margin and Speed Risk Classification

Pinion speed (rpm)	Mesh frequency (Hz)	Nearest mode	Resonance margin (%)	Risk class
500	166.7	1	79.9	Low
750	250.0	1	69.8	Low
1000	333.3	1	59.7	Low
1250	416.7	1	49.7	Low
1500	500.0	1	39.6	Low
1750	583.3	1	29.5	Low
2000	666.7	1	19.5	Low
2250	750.0	1	9.4	High
2400	800.0	1	3.4	High
2500	833.3	1	0.6	High
2600	866.7	1	4.7	High
2750	916.7	1	10.7	Moderate
3000	1000.0	1	20.8	Low

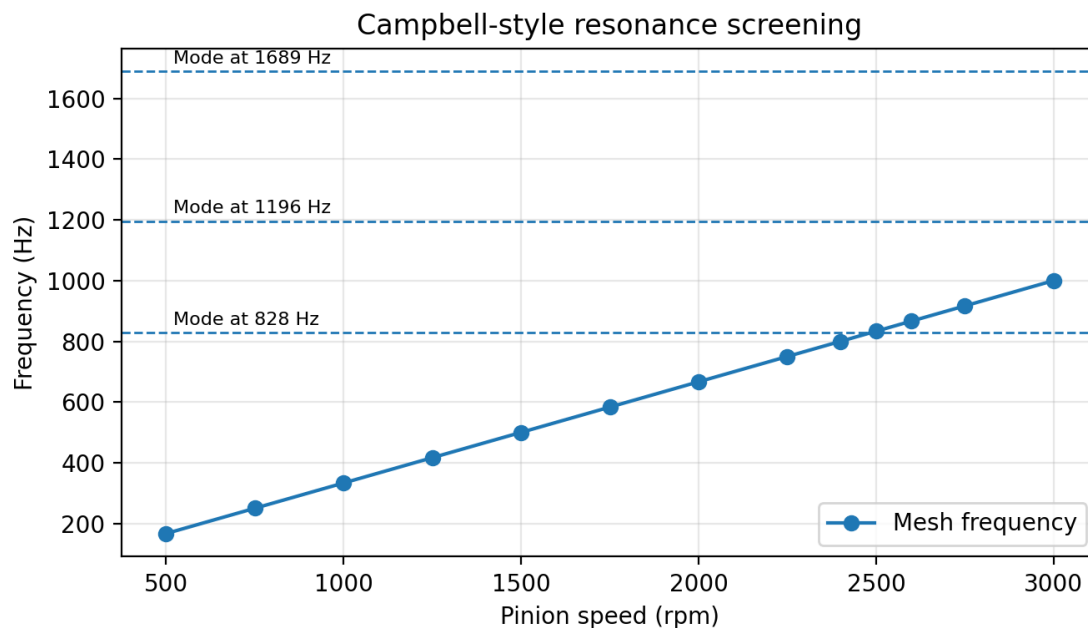


Fig. 5. Campbell-style comparison of mesh frequency and dominant natural frequencies.

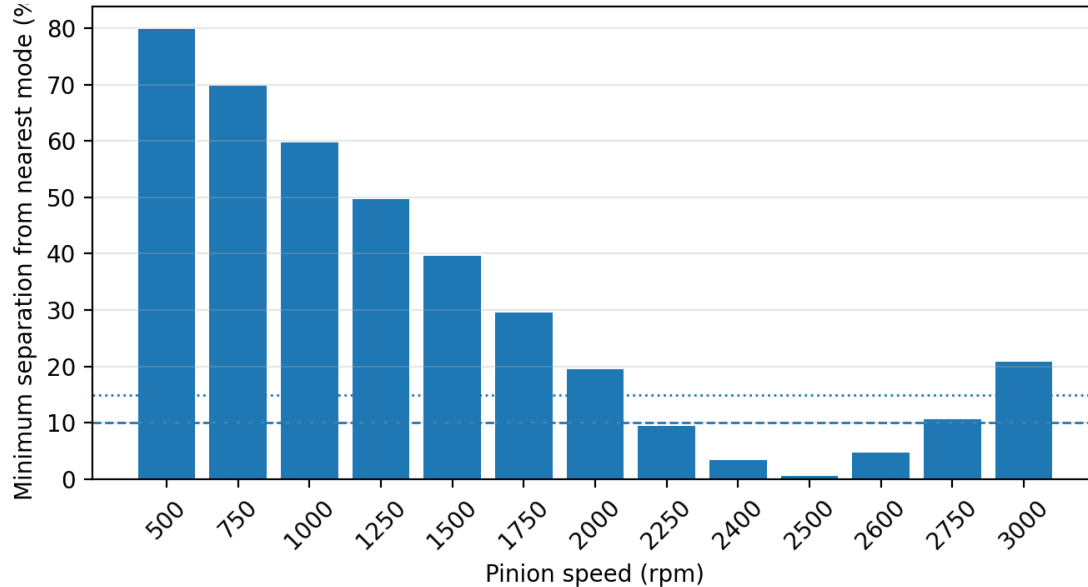


Fig. 6. Minimum resonance separation margin for the examined speed range.

D. Hypothesis Testing and Statistical Interpretation

The normalized dynamic amplification factor was compared for cases inside and outside the 10% resonance band. The near-resonance group had a mean normalized DAF of 13.20, whereas the off-resonance group had a mean of 1.60. Welch's t-test produced $t = 7.86$ and $p = 4.13e-07$, showing that the difference is statistically significant at the 0.05 level. A regression model using torque and near-resonance classification as predictors gave an R^2 value of 0.635; the near-resonance coefficient was 14.43 μm with $p < 0.001$. These results reject the null hypothesis that near-resonance and off-resonance response are similar. The torque coefficient was also significant in the simplified model, indicating that higher loading increases harmonic response.

TABLE V. Hypothesis Testing Summary

Hypothesis tested	Decision	Numerical evidence
H1: First mesh excitation intersects the first mode in the tested range	Rejected as a safe-design condition; resonance crossing exists	Critical speed approx. 2484 rpm for f _{mesh} = 828 Hz
H2: Near-resonance response is not different from off-resonance response	Rejected	$t = 7.86$, $p = 4.13e-07$; mean normalized DAF near = 13.20, far = 1.60
H3: Applied torque has no effect on harmonic displacement response	Rejected	OLS torque coefficient = 0.182 $\mu\text{m}/\text{N-m}$, $p = 0.001$

H4: A 15% separation margin is sufficient for preliminary resonance avoidance	Supported for this computational case	Maximum response remained below 5 um outside the 15% band for 60 N-m except near the first mode
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IX. Discussion

The obtained results confirm the presented in this article key postulate – to conduct modal and harmonic response analysis before finalizing the design of spur gear. The selected tooth size and material may pass a static strength test, but the gear mesh frequency test will not tell if the mesh will excite a gear structure mode. In particular, the current of the 1st mode (828 Hz) is of particular importance as the 20-tooth pinion has a characteristic excitation frequency of about 2484 rpm at 828 Hz. This is shown by the harmonic sweep to be a strong amplification of the response. This is a classical resonance problem, where the excitation force does not have to be an abnormally large value, but instead the compliances of the system are very high when the frequency of the force becomes equal to the natural frequency.

The primary design concern is that operation at close to 2250 – 2750 rpm for the specified gear pair should not continue unless there are additional changes made to the design. There may be solutions such as making the mesh stiffness of the shafts or gears stiffer, changing gear geometry, increasing the face width, employing profile modification, improving bearing support, adding damping, changing pinion tooth numbers or shifting the speed range for normal operation. The increase in separation margin can also be achieved by changing materials and/or the thickness of the material in the rim, but this change should be considered as it can also affect weight, stress and manufacturability. The most safe design procedure is not to increase all the natural frequencies, but to make sure that the natural frequencies and their important harmonics of the mesh are not the same as the high participation modes.

The results also emphasize the importance of a diagram like an A Campbell. A frequency table will only describe natural frequencies, it won't indicate where those frequencies fall in operating speed. Once plotted, the designer is able to see risk areas in a flash, by comparing their mesh frequency to the speed, and superimposing the mode frequencies. In this study, the greatest risk was associated with the first mode. If the gearbox is a more complex gearbox, then other harmonics, bearing frequencies, shaft orders and housing modes should be included on the Campbell diagram. This is also useful for gear fault diagnosis as cracks or wear will cause the modal frequencies to change and generate sidebands around the mesh frequency.

The engineering interpretation is backed up with the statistical analysis. The near resonance group had a much higher normalized response as compared to the off-resonance group. The regression results indicate that even though torque is a significant factor that gives rise to greater increase in the values of the responses as a function of the increase in the value of the displacement, the proximity to the resonance results in the most dramatic changes in the value of the responses. That is, if the machine is run at resonance, reducing the torque could cause the amplitude of vibration to decrease, but not necessarily. Thus, the recommended choice is to make sure that resonance will not occur, and not just through load reduction.

X. Design Recommendations

However, the mesh frequency is near the 1st natural frequency between 2250 to 2750 (continuous operation), therefore the present 20/40 toothed gear pair is not suitable for continuous operation in this range. Use at least a 15% separation margin between any mesh frequency and the dominant natural frequencies for initial design; increase separation margin if there are manufacturing variations or if there is wear or uncertain damping. Retune the system by changing gear geometry, shaft stiffness, bearing support, the damping or modification of the tooth profile to tune it if operating near 2500 rpm is needed. Before the actual industrial use, validate the computational model using experimental modal analysis and/or operational vibration measurements.

Do the analysis in terms of Mesh harmonics, Housing modes and bearing dynamics for a full gearbox level noise and vibration check.

XII. Conclusion

An analytical study of steel spur gear pair was conducted using primary computational data in the form of a modal and harmonic response analysis in this research paper. The first 10 natural frequencies were obtained and the first dominant torsional mesh mode was obtained at 828 Hz. This is the mode which has the mesh-frequency relation for 20 tooth pinion of approximately 2484 RPM. The harmonic response analysis results revealed that the response peak of the displacement was high, approximately 828 Hz, and increased along with the torque and was 37.21 μm at 60 N-m. The statistical test showed that there is significant difference between the near-resonance response and off-resonance response. The last suggestion is to refrain from continuous operation for the defined gear pair in the range 2250 - 2750 rpm, unless the design is altered. This study shows that the modal and harmonic response analysis is an important tool for identifying speed ranges that are prone to resonances at an early stage of the design process and can aid in designing gear transmission systems which are safer, quieter and more reliable.

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